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Nonlinear Control

Hassan K. Khalil

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To my parents
Mohamed and Fat-hia

and my grandchildren
Maryam, Tariq, Aya, and Tessneem

Contents

Preface

This book emerges from my earlier book *Nonlinear Systems*, but it is not a fourth edition of it nor a replacement for it. Its mission and organization are different from *Nonlinear Systems*. While *Nonlinear Systems* was intended as a reference and a text on nonlinear system analysis and its application to control, this book is intended as a text for a first course on nonlinear control that can be taught in one semester (forty lectures). The writing style is intended to make it accessible to a wider audience without compromising the rigor, which is a characteristic of *Nonlinear Systems*. Proofs are included only when they are needed to understand the material; otherwise references are given. In a few cases when it is not convenient to find the proofs in the literature, they are included in the Appendix. With the size of this book about half that of *Nonlinear Systems*, naturally many topics had to be removed. This is not a reflection on the importance of these topics; rather it is my judgement of what should be presented in a first course. Instructors who used *Nonlinear Systems* may disagree with my decision to exclude certain topics; to them I can only say that those topics are still available in *Nonlinear Systems* and can be integrated into the course.

An electronic solution manual is available to instructors from the publisher, not the author. The instructors will also have access to Simulink models of selected exercises. The Instructor Resource Center (IRC) for this book (www.pearsonglobal editions.com/khalil) contains the solution manual, the Simulink models of selected examples and the pdf slides of the course. To gain access to the IRC, please contact your local Pearson sales representative.

The book was typeset using LaTeX. Computations were done using MATLAB and Simulink. The figures were generated using MATLAB or the graphics tool of LaTeX.

I am indebted to many colleagues, students, and readers of *Nonlinear Systems*, and reviewers of this manuscript whose feedback was a great help in writing this book. I am grateful to Michigan State University for an environment that allowed me to write the book, and to the National Science Foundation for supporting my research on nonlinear feedback control.

Hassan Khalil

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Chapter 1

Introduction

The chapter starts in Section 1.1 with a definition of the class of nonlinear state models that will be used throughout the book. It briefly discusses three notions associated with these models: existence and uniqueness of solutions, change of variables, and equilibrium points. Section 1.2 explains why nonlinear tools are needed in the analysis and design of nonlinear systems. Section 1.3 is an overview of the next twelve chapters.

1.1 Nonlinear Models

We shall deal with dynamical systems, modeled by a finite number of coupled first-order ordinary differential equations:

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_m)
\end{aligned}
$$

where $\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t and u_1, u_2, …, u_m are input variables. We call x_1, x_2, …, x_n the state variables. They represent the memory that the dynamical system has of its past. We usually use vector notation to write these equations in a compact form. Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

and rewrite the n first-order differential equations as one n-dimensional first-order vector differential equation

$$\dot{x} = f(t,x,u) \tag{1.1}$$

We call (1.1) the state equation and refer to x as the *state* and u as the *input*. Sometimes another equation,

$$y = h(t,x,u) \tag{1.2}$$

is associated with (1.1), thereby defining a q-dimensional *output* vector y that comprises variables of particular interest, like variables that can be physically measured or variables that are required to behave in a specified manner. We call (1.2) the output equation and refer to equations (1.1) and (1.2) together as the state-space model, or simply the state model. Several examples of nonlinear state models are given in Appendix A and in exercises at the end of this chapter. For linear systems, the state model (1.1)–(1.2) takes the special form

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

Sometimes we consider a special case of (1.1) without explicit presence of an input u, the so-called unforced state equation:

$$\dot{x} = f(t,x) \tag{1.3}$$

This case arises if there is no external input that affects the behavior of the system, or if the input has been specified as a function of time, $u = \gamma(t)$, a feedback function of the state, $u = \gamma(x)$, or both, $u = \gamma(t,x)$. Substituting $u = \gamma$ in (1.1) eliminates u and yields an unforced state equation.

In dealing with equation (1.3), we shall typically require the function $f(t,x)$ to be piecewise continuous in t and locally Lipschitz in x over the domain of interest. For a fixed x, the function $f(t,x)$ is piecewise continuous in t on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f may have finite-jump discontinuities.

This allows for cases where $f(t, x)$ depends on an input $u(t)$ that may experience step changes with time. A function $f(t, x)$, defined for $t \in J \subset R$, is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0, r)$ of x_0, defined by $N(x_0, r) = \{\|x - x_0\| < r\}$, and a positive constant L such that $f(t, x)$ satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \ \leq \ L\|x - y\| \tag{1.4}$$

for all $t \in J$ and all $x, y \in N(x_0, r)$, where

$$\|x\| = \sqrt{x^T x} = \sqrt{x_1^2 + \cdots + x_n^2}$$

A function $f(t, x)$ is locally Lipschitz in x on a domain (open and connected set) $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$. It is Lipschitz on a set W if it satisfies (1.4) for all points in W, with the same Lipschitz constant L. A locally Lipschitz function on a domain D is not necessarily Lipschitz on D, since the Lipschitz condition may not hold uniformly (with the same constant L) for all points in D. However, a locally Lipschitz function on a domain D is Lipschitz on every compact (closed and bounded) subset of D. A function $f(t, x)$ is *globally Lipschitz* if it is Lipschitz on R^n.

When $n = 1$ and f depends only on x, the Lipschitz condition can be written as

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

which implies that on a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L. Therefore, any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point. For example, any discontinuous function is not locally Lipschitz at the points of discontinuity. As another example, the function $f(x) = x^{1/3}$ is not locally Lipschitz at $x = 0$ since $f'(x) = (1/3)x^{-2/3} \to \infty$ as $x \to 0$. On the other hand, if $f'(x)$ is continuous at a point x_0 then $f(x)$ is locally Lipschitz at the same point because continuity of $f'(x)$ ensures that $|f'(x)|$ is bounded by a constant k in a neighborhood of x_0; which implies that $f(x)$ satisfies the Lipschitz condition (1.4) over the same neighborhood with $L = k$.

More generally, if for t in an interval $J \subset R$ and x in a domain $D \subset R^n$, the function $f(t, x)$ and its partial derivatives $\partial f_i/\partial x_j$ are continuous, then $f(t, x)$ is locally Lipschitz in x on D.[1] If $f(t, x)$ and its partial derivatives $\partial f_i/\partial x_j$ are continuous for all $x \in R^n$, then $f(t, x)$ is globally Lipschitz in x if and only if the partial derivatives $\partial f_i/\partial x_j$ are globally bounded, uniformly in t, that is, their absolute values are bounded for all $t \in J$ and $x \in R^n$ by constants independent of (t, x).[2]

[1] See [74, Lemma 3.2] for the proof of this statement.
[2] See [74, Lemma 3.3] for the proof of this statement.

Example 1.1 The function

$$f(x) = \begin{bmatrix} -x_1 + x_1x_2 \\ x_2 - x_1x_2 \end{bmatrix}$$

is continuously differentiable on R^2. Hence, it is locally Lipschitz on R^2. It is not globally Lipschitz since $\partial f_1/\partial x_2$ and $\partial f_2/\partial x_1$ are not uniformly bounded on R^2. On any compact subset of R^2, f is Lipschitz. Suppose we are interested in calculating a Lipschitz constant over the set $W = \{|x_1| \leq a,\ |x_2| \leq a\}$. Then,

$$|f_1(x) - f_1(y)| \leq |x_1 - y_1| + |x_1x_2 - y_1y_2|$$
$$|f_2(x) - f_2(y)| \leq |x_2 - y_2| + |x_1x_2 - y_1y_2|$$

Using the inequalities

$$|x_1x_2 - y_1y_2| = |x_1(x_2 - y_2) + y_2(x_1 - y_1)| \leq a|x_2 - y_2| + a|x_1 - y_1|$$
$$|x_1 - y_1|\ |x_2 - y_2| \leq \tfrac{1}{2}|x_1 - y_1|^2 + \tfrac{1}{2}|x_2 - y_2|^2$$

we obtain

$$\|f(x) - f(y)\|^2 = |f_1(x) - f_1(y)|^2 + |f_2(x) - f_2(y)|^2 \leq (1 + 2a)^2\|x - y\|^2$$

Therefore, f is Lipschitz on W with the Lipschitz constant $L = 1 + 2a$. △

Example 1.2 The function

$$f(x) = \begin{bmatrix} x_2 \\ -\text{sat}(x_1 + x_2) \end{bmatrix}$$

is not continuously differentiable on R^2. Using the fact that the saturation function $\text{sat}(\cdot)$ satisfies $|\text{sat}(\eta) - \text{sat}(\xi)| \leq |\eta - \xi|$, we obtain

$$\begin{aligned} \|f(x) - f(y)\|^2 &\leq (x_2 - y_2)^2 + (x_1 + x_2 - y_1 - y_2)^2 \\ &\leq (x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2 \end{aligned}$$

Using the inequality

$$a^2 + 2ab + 2b^2 = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \lambda_{\max}\left\{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right\} \times \left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\|^2$$

we conclude that

$$\|f(x) - f(y)\| \leq \sqrt{2.618}\ \|x - y\|, \qquad \forall\ x, y \in R^2$$

Here we have used a property of positive semidefinite symmetric matrices; that is, $x^TPx \leq \lambda_{\max}(P)\ x^Tx$, for all $x \in R^n$, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of P. A more conservative (larger) Lipschitz constant will be obtained if we use the more conservative inequality

$$a^2 + 2ab + 2b^2 \leq 2a^2 + 3b^2 \leq 3(a^2 + b^2)$$

resulting in a Lipschitz constant $L = \sqrt{3}$. △

The local Lipschitz property of $f(t,x)$ ensures local existence and uniqueness of the solution of the state equation (1.3), as stated in the following lemma.[3]

Lemma 1.1 *Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$.* $\diamond$

Without the local Lipschitz condition, we cannot ensure uniqueness of the solution. For example, the state equation $\dot{x} = x^{1/3}$, whose right-hand side function is continuous but not locally Lipschitz at $x = 0$, has $x(t) = (2t/3)^{3/2}$ and $x(t) \equiv 0$ as two different solutions when the initial state is $x(0) = 0$.

Lemma 1.1 is a local result because it guarantees existence and uniqueness of the solution over an interval $[t_0, t_0 + \delta]$, but this interval might not include a given interval $[t_0, t_1]$. Indeed the solution may cease to exist after some time.

Example 1.3 In the one-dimensional system $\dot{x} = -x^2$, the function $f(x) = -x^2$ is locally Lipschitz for all x. Yet, when we solve the equation with $x(0) = -1$, the solution $x(t) = 1/(t-1)$ tends to $-\infty$ as $t \to 1$. $\triangle$

The phrase "finite escape time" is used to describe the phenomenon that a solution escapes to infinity at finite time. In Example 1.3, we say that the solution has a finite escape time at $t = 1$.

In the forthcoming Lemmas 1.2 and 1.3,[4] we shall give conditions for global existence and uniqueness of solutions. Lemma 1.2 requires the function f to be globally Lipschitz, while Lemma 1.3 requires f to be only locally Lipschitz, but with an additional requirement that the solution remains bounded. Note that the function $f(x) = -x^2$ of Example 1.3 is locally Lipschitz for all x but not globally Lipschitz because $f'(x) = -2x$ is not globally bounded.

Lemma 1.2 *Let $f(t,x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t,x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$.* $\diamond$

The global Lipschitz condition is satisfied for linear systems of the form

$$\dot{x} = A(t)x + g(t)$$

when $\|A(t)\| \le L$ for all $t \ge t_0$, but it is a restrictive condition for general nonlinear systems. The following lemma avoids this condition.

Lemma 1.3 *Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \ge t_0$ and all x in a domain $D \subset R^n$. Let W be a compact (closed and bounded) subset of D, $x_0 \in W$, and suppose it is known that every solution of*

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

lies entirely in W. Then, there is a unique solution that is defined for all $t \ge t_0$. $\diamond$

[3]See [74, Theorem 3.1] for the proof of Lemma 1.1. See [56, 62, 95] for a deeper look into existence and uniqueness of solutions, and the qualitative behavior of nonlinear differential equations.

[4]See [74, Theorem 3.2] and [74, Theorem 3.3] for the proofs of Lemmas 1.2 and 1.3, respectively.

The trick in applying Lemma 1.3 is in checking the assumption that every solution lies in a compact set without solving the state equation. We will see in Chapter 3 that Lyapunov's method for stability analysis provides a tool to ensure this property. For now, let us illustrate the application of the lemma by an example.

Example 1.4 Consider the one-dimensional system

$$\dot{x} = -x^3 = f(x)$$

The function $f(x)$ is locally Lipschitz on R, but not globally Lipschitz because $f'(x) = -3x^2$ is not globally bounded. If, at any instant of time, $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative and $x(t)$ will be decreasing. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive and $x(t)$ will be increasing. Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{|x| \leq |a|\}$. Thus, we conclude by Lemma 1.3 that the equation has a unique solution for all $t \geq 0$. $\triangle$

A special case of (1.3) arises when the function f does not depend explicitly on t; that is,

$$\dot{x} = f(x)$$

in which case the state equation is said to be *autonomous* or *time invariant*. The behavior of an autonomous system is invariant to shifts in the time origin, since changing the time variable from t to $\tau = t - a$ does not change the right-hand side of the state equation. If the system is not autonomous, then it is called *nonautonomous* or *time varying*.

More generally, the state model (1.1)–(1.2) is said to be *time invariant* if the functions f and h do not depend explicitly on t; that is,

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

If either f or h depends on t, the state model is said to be time varying. A time-invariant state model has a time-invariance property with respect to shifting the initial time from t_0 to $t_0 + a$, provided the input waveform is applied from $t_0 + a$ instead of t_0. In particular, let $(x(t), y(t))$ be the response for $t \geq t_0$ to initial state $x(t_0) = x_0$ and input $u(t)$ applied for $t \geq t_0$, and let $(\tilde{x}(t), \tilde{y}(t))$ be the response for $t \geq t_0 + a$ to initial state $\tilde{x}(t_0 + a) = \tilde{x}_0$ and input $\tilde{u}(t)$ applied for $t \geq t_0 + a$. Now, take $\tilde{x}_0 = x_0$ and $\tilde{u}(t) = u(t - a)$ for $t \geq t_0 + a$. By changing the time variable from t to $t - a$ it can be seen that $\tilde{x}(t) = x(t - a)$ and $\tilde{y}(t) = y(t - a)$ for $t \geq t_0 + a$. Therefore, for time-invariant systems, we can, without loss of generality, take the initial time to be $t_0 = 0$.

A useful analysis tool is to transform the state equation from the x-coordinates to the z-coordinates by the change of variables $z = T(x)$. For linear systems, the change of variables is a similarity transformation $z = Px$, where P is a nonsingular matrix. For a nonlinear change of variables, $z = T(x)$, the map T must be invertible; that is, it must have an inverse map $T^{-1}(\cdot)$ such that $x = T^{-1}(z)$ for all $z \in T(D)$, where D is the domain of T. Moreover, because the derivatives of z and x should be continuous, we require both $T(\cdot)$ and $T^{-1}(\cdot)$ to be continuously differentiable. A continuously differentiable map with a continuously differentiable inverse is known as a *diffeomorphism.* A map $T(x)$ is a *local diffeomorphism* at a point x_0 if there is a neighborhood N of x_0 such that T restricted to N is a diffeomorphism on N. It is a global diffeomorphism if it is a diffeomorphism on R^n and $T(R^n) = R^n$. The following lemma gives conditions on a map $z = T(x)$ to be a local or global diffeomorphism using the Jacobian matrix $[\partial T/\partial x]$, which is a square matrix whose (i,j) element is the partial derivative $\partial T_i/\partial x_j$.[5]

Lemma 1.4 *The continuously differentiable map $z = T(x)$ is a local diffeomorphism at x_0 if the Jacobian matrix $[\partial T/\partial x]$ is nonsingular at x_0. It is a global diffeomorphism if and only if $[\partial T/\partial x]$ is nonsingular for all $x \in R^n$ and T is proper; that is,* $\lim_{\|x\|\to\infty} \|T(x)\| = \infty$. ◇

Example 1.5 In Section A.4 two different models of the negative resistance oscillator are given, which are related by the change of variables

$$z = T(x) = \begin{bmatrix} -h(x_1) - x_2/\varepsilon \\ x_1 \end{bmatrix}$$

Assuming that $h(x_1)$ is continuously differentiable, the Jacobian matrix is

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \\ \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -h'(x_1) & -1/\varepsilon \\ 1 & 0 \end{bmatrix}$$

Its determinant, $1/\varepsilon$, is positive for all x. Moreover, $T(x)$ is proper because

$$\|T(x)\|^2 = [h(x_1) + x_2/\varepsilon]^2 + x_1^2$$

which shows that $\lim_{\|x\|\to\infty} \|T(x)\| = \infty$. In particular, if $|x_1| \to \infty$, so is $\|T(x)\|$. If $|x_1|$ is finite while $|x_2| \to \infty$, so is $[h(x_1) + x_2/\varepsilon]^2$ and consequently $\|T(x)\|$. △

Equilibrium points are important features of the state equation. A point x^* is an equilibrium point of $\dot{x} = f(t,x)$ if the equation has a constant solution $x(t) \equiv x^*$. For the time-invariant system $\dot{x} = f(x)$, equilibrium points are the real solutions of

$$f(x) = 0$$

An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points. The linear system $\dot{x} = Ax$ has an isolated equilibrium point at $x = 0$ when A is nonsingular or a continuum of equilibrium points in the null space of A when A is singular. It

[5]The proof of the local result follows from the inverse function theorem [3, Theorem 7-5]. The proof of the global results can be found in [117] or [150].

cannot have multiple isolated equilibrium points, for if x_a and x_b are two equilibrium points, then by linearity any point on the line $\alpha x_a + (1-\alpha)x_b$ connecting x_a and x_b will be an equilibrium point. A nonlinear state equation can have multiple isolated equilibrium points. For example, the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2$$

has equilibrium points at $(x_1 = n\pi, x_2 = 0)$ for $n = 0, \pm 1, \pm 2, \cdots$.

1.2 Nonlinear Phenomena

The powerful analysis tools for linear systems are founded on the basis of the *superposition principle*. As we move from linear to nonlinear systems, we are faced with a more difficult situation. The superposition principle no longer holds, and analysis involves more advanced mathematics. Because of the powerful tools we know for linear systems, the first step in analyzing a nonlinear system is usually to linearize it, if possible, about some nominal operating point and analyze the resulting linear model. This is a common practice in engineering, and it is a useful one. However, there are two basic limitations of linearization. First, since linearization is an approximation in the neighborhood of an operating point, it can only predict the "local" behavior of the nonlinear system in the vicinity of that point. It cannot predict the "nonlocal" behavior far from the operating point and certainly not the "global" behavior throughout the state space. Second, the dynamics of a nonlinear system are much richer than the dynamics of a linear system. There are "essentially nonlinear phenomena" that can take place only in the presence of nonlinearity; hence, they cannot be described or predicted by linear models. The following are examples of essentially nonlinear phenomena:

- *Finite escape time.* The state of an unstable linear system goes to infinity as time approaches infinity; a nonlinear system's state, however, can go to infinity in finite time.

- *Multiple isolated equilibria.* A linear system can have only one isolated equilibrium point; thus, it can have only one steady-state operating point that attracts the state of the system irrespective of the initial state. A nonlinear system can have more than one isolated equilibrium point. The state may converge to one of several steady-state operating points, depending on the initial state of the system.

- *Limit cycles.* For a linear time-invariant system to oscillate, it must have a pair of eigenvalues on the imaginary axis, which is a nonrobust condition that is almost impossible to maintain in the presence of perturbations. Even if we do so, the amplitude of oscillation will be dependent on the initial state. In real life, stable oscillation must be produced by nonlinear systems. There are nonlinear systems that can go into oscillation of fixed amplitude and frequency, irrespective of the initial state. This type of oscillation is known as limit cycles.

- *Subharmonic, harmonic, or almost-periodic oscillations.* A stable linear system under a periodic input produces a periodic output of the same frequency. A nonlinear system under periodic excitation can oscillate with frequencies that are submultiples or multiples of the input frequency. It may even generate an almost-periodic oscillation, an example of which is the sum of periodic oscillations with frequencies that are not multiples of each other.

- *Chaos.* A nonlinear system can have a more complicated steady-state behavior that is not equilibrium, periodic oscillation, or almost-periodic oscillation. Such behavior is usually referred to as chaos. Some of these chaotic motions exhibit randomness, despite the deterministic nature of the system.

- *Multiple modes of behavior.* It is not unusual for two or more modes of behavior to be exhibited by the same nonlinear system. An unforced system may have more than one limit cycle. A forced system with periodic excitation may exhibit harmonic, subharmonic, or more complicated steady-state behavior, depending upon the amplitude and frequency of the input. It may even exhibit a discontinuous jump in the mode of behavior as the amplitude or frequency of the excitation is smoothly changed.

In this book, we encounter only the first three of these phenomena.[6] The phenomenon of finite escape time has been already demonstrated in Example 1.3, while multiple equilibria and limit cycles will be introduced in the next chapter.

1.3 Overview of the Book

Our study of nonlinear control starts with nonlinear analysis tools that will be used in the analysis and design of nonlinear control systems. Chapter 2 introduces phase portraits for the analysis of two-dimensional systems and illustrates some essentially nonlinear phenomena. The next five chapters deal with stability analysis of nonlinear systems. Stability of equilibrium points is defined and studied in Chapter 3 for time-invariant systems. After presenting some preliminary results for linear systems, linearization, and one-dimensional systems, the technique of Lyapunov stability is introduced. It is the main tool for stability analysis of nonlinear systems. It requires the search for a scalar function of the state, called Lyapunov function, such that the function and its time derivative satisfy certain conditions. The challenge in Lyapunov stability is the search for a Lyapunov function. However, by the time we reach the end of Chapter 7, the reader would have seen many ideas and examples of how to find Lyapunov functions. Additional ideas are given

[6] To read about forced oscillation, chaos, bifurcation, and other important topics, consult [52, 55, 136, 146].

in Appendix C. Chapter 4 extends Lyapunov stability to time-varying systems and shows how it can be useful in the analysis of perturbed system. This leads into the notion of input-to-state stability. Chapter 5 deals with a special class of systems that dissipates energy. One point we emphasize is the connection between passivity and Lyapunov stability. Chapter 6 looks at input-output stability and shows that it can be established using Lyapunov functions. The tools of Chapters 5 and 6 are used in Chapter 7 to derive stability criteria for the feedback connection of two stable systems.

The next six chapters deal with nonlinear control. Chapter 8 presents some special nonlinear forms that play significant roles in the design of nonlinear controllers. Chapters 9 to 13 deal with nonlinear control problems, including nonlinear observers. The nonlinear control techniques we are going to study can be categorized into five different approaches to deal with nonlinearity. These are:

- Approximate nonlinearity
- Compensate for nonlinearity
- Dominate nonlinearity
- Use intrinsic properties
- Divide and conquer

Linearization is the prime example of approximating nonlinearities. Feedback linearization that cancels nonlinearity is an example of nonlinearity compensation. Robust control techniques, which are built around the classical tool of high-gain feedback, dominate nonlinearities. Passivity-based control is an example of a technique that takes advantage of an intrinsic property of the system. Because the complexity of a nonlinear system grows rapidly with dimension, one of the effective ideas is to decompose the system into lower-order components, which might be easier to analyze and design, then build up back to the original system. Backstepping is an example of this divide and conquer approach.

Four appendices at the end of the book give examples of nonlinear state models, mathematical background, procedures for constructing composite Lyapunov functions, and proofs of some results. The topics in this book overlap with topics in some excellent textbooks, which can be consulted for further reading. The list includes [10, 53, 63, 66, 92, 118, 129, 132, 144]. The main source for the material in this book is [74], which was prepared using many references. The reader is advised to check the Notes and References section of [74] for a detailed account of these references.

1.4 Exercises

1.1 A general mathematical model that describes the system with n state variables, m input variables and p output variables is given by

$$\dot{x}_1 = f_1(x_1, \dots x_n, u_1, \dots u_m), \qquad \dot{x}_n = f_n(x_1, \dots x_n, u_1, \dots u_m)$$
$$y_1 = g_1(x_1, \dots x_n, u_1, \dots u_m) \qquad y_p = g_p(x_1, \dots x_n, u_1, \dots u_m)$$

where u is the input and y is the output. Linearise the model at an equilibrium point $\bar{x} = [\bar{x}_1, \bar{x}_2, \dots \bar{x}_n]^T$ and $\bar{u} = [\bar{u}_1, \bar{u}_2, \dots \bar{u}_m]^T$. Find the state space model.